Parametric Response of Crooked Thin-Walled Columns

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The nonlinear partial differential equations of a thin-walled open cross section column which is not straight in the initial unstressed state are developed. The column is loaded by a harmonic longitudinal load, and is simply supported. Variables are separated by assuming the spatial mode to be identical to the fundamental buckling mode. Geometric nonlinearities provide coupling in the case of columns with doubly symmetric cross sections which is not predicted by linear theory. The two bending modes are coupled by the twisting mode. The stability analysis can be performed by considering only the Mathieu terms of the three equations, and the unstable region can be minimized when the parameters are adjusted so that the Mathieu terms for the three equations are identical. However, for this case, it is shown that virtually the entire region is unstable. This paradox is attributed to the assumption of zero damping and confirmed by solutions with linear viscous damping included. Analog computer solutions demonstrate the importance of considering geometric nonlinearities.

Nomenclature

A,B,C = initial unstressed displacement functions (crookedness)
A_s = cross-sectional area
a,b = initial dimensionless lateral displacements at the midpoint of the column in unstressed state
C_w = warping coefficient
c = initial angular displacement at the midpoint of the column in unstressed state
E = modulus of elasticity
G = shearing modulus of elasticity
I_x,I_y = principal moments of inertia of the cross section
I_1^* = dimensionless ratio of principal moments of inertia, I_x/I_y
I_2^* = dimensionless torsional stiffness, $(GA_sJ_eL^2/\pi^2EI_yJ_0) + (C_wA_s/I_yJ_0)$
J_e = St Venant's torsional constant
J_0 = polar moment of inertia with respect to origin
L = length of column
$M_{\bar{x}},M_{\bar{y}},M_{\bar{z}}$ = moments about the $\bar{x}$, $\bar{y}$, $\bar{z}$ axes
m = mass per unit length of the column
P = axial time dependent load
P_0 = axial static load
P_1 = axial dynamic overload
P^* = dimensionless axial load, PL^2/π^2EI_y
r_x,r_y = radii of gyration of the cross section with respect to x and y axes
T = dimensionless time, θt
t = time
u,v = bending deformations from the initially straight configuration
X,Y = temporal mode displacement amplitudes as functions of T
x,y,z = orthogonal axes fixed at the end of the column
$\bar{x},\bar{y},\bar{z}$ = orthogonal principal axes positioned in the section a distance z from the end of the column
α = frequency parameter, $\theta/[2\omega_1(1 - P_0^*)^{1/2}]$
β = loading parameter, $P_1^*/[2(1 - P_0^*)]$
θ = frequency of excitation in radians per sec
Φ = temporal mode angular displacement amplitude as a function of T
φ = total angular displacement about the z axis from the untwisted configuration
ω_1 = fundamental frequency of vibration $\pi^2(EI_y)^{1/2}/L^2m^{1/2}$
Ω = ratio of the driving frequency to the natural frequency of the unloaded column, θ/ω_1

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Introduction

THERE exists a continuing interest in parametric oscillations. This type of structural vibrations occur in many places including supports for rotating machinery, aircraft structures and certain rotating satellites. Ordinary resonance occurs when the frequency of the exciting force is equal to one of the natural frequencies of the physical system. Parametric resonance associated with a particular vibrational mode can occur when the driving frequency is higher than or lower than the natural frequency corresponding to that mode. A comprehensive review of the subject of parametric response of structures is given by Evan-Iwanowski,[3] and the book by Bolotin[1] is a classic work on the parametric stability of elastic systems. Many investigators have considered the in-plane vibration of columns under parametric loading and have considered longitudinal inertia effects, initially curved but unstressed shapes, and nonlinear effects of damping and elasticity (large deflections). These investigations include stability analyses as well as expressions for approximate steady-state amplitudes. However, only limited investigations exist for three-dimensional parametric column vibrations which is the concern of this paper.

The linear equations for coupled vibrations of thin-walled columns of open cross sections are due to Vlasov[11] and Gere.[5] From these equations it is shown that the transverse vibrations in two perpendicular planes and the torsional vibrations are dynamically coupled if the section is asymmetrical. For the case of a doubly symmetric cross section these equations show that coupling of the twisting and bending motion does not exist. Bolotin discusses the three linear differential equations representing parametric vibrations of thin-walled columns and derives equations for the primary regions of instability subject to certain simplifying assumptions. The purpose herein is to show that geometric nonlinearities associated with small deflection theory can cause significant coupling of the bending and twisting motion in the case of a

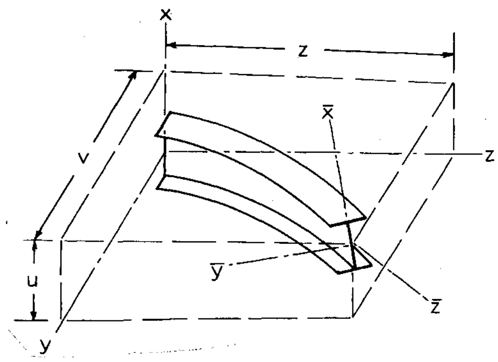

Fig. 1 Beam segment and coordinate systems.

column with a doubly symmetric cross section. The investigation consists of obtaining the dynamic response of uniform slender elastic columns of thin-walled open cross sections subjected to an axial static load plus a harmonic overload. Columns with initial unstressed crookedness (nonstraight and initial twist) and with low torsional stiffness are considered.

Equations of Motion

Thin-walled structural columns are bodies which have the form of long prismatic shells. These members are distinguished by the fact that their three dimensions are all of a different order of magnitude. For the structural elements considered herein it is assumed that the dimensions of the cross section are small, when compared to the length of the column, but are large when compared to the thickness of the walls of the column. If an axial force is applied to a thin-walled column of open cross section, vibrations accompanied by bending in two planes together with twisting may occur. Figure 1 shows a thin-walled column segment with an open cross section subjected to an axial load P. The orthogonal axes x, y, and z are fixed at one end of the column and positioned in such a manner that the longitudinal axis, z, passes through the centroid of the doubly symmetric cross section, and x and y are the principal axes of the cross section. A second system of axes, $\bar{x}$, $\bar{y}$, and $\bar{z}$, is positioned at a distance z from the end of the column and is fixed to the cross section so that $\bar{x}$ and $\bar{y}$ are its principal axes. Thus, the $\bar{x}$, $\bar{y}$, $\bar{z}$ axes translate laterally the distance u and v as shown and rotate with the plane of the cross section. The rotations are $\partial v/\partial z$, $\partial u/\partial z$ and φ respectively about the axes $\bar{x}$, $\bar{y}$, and $\bar{z}$. Internal bending moments and the twisting moment due to the axial load P are

$$M_{\bar{x}} = P(u\varphi - v), \quad M_{\bar{y}} = P(u + v\varphi)$$
$$M_{\bar{z}} = P(u\partial v/\partial z - v\partial u/\partial z) \tag{1}$$

The moment curvature equations and the nonuniform torsion equation[11] written for the case of an initially crooked column are

$$EI_y\partial^2(u - A)/\partial z^2 = - M_{\bar{y}} \tag{2a}$$

$$EI_x\partial^2(v - B)/\partial z^2 = M_{\bar{x}} \tag{2b}$$

$$EC_w\partial^3(\varphi - C)/\partial z^3 - GJ_e\partial(\varphi - C)/\partial z + (P/A_s)J_0\partial\varphi/\partial z = M_{\bar{z}} \tag{2c}$$

in which A, B, and C are functions which define the initial crookedness of the column. Equations (2a) and (2b) are converted to equations of motion by differentiating twice with respect to z and adding the inertia forces. For a doubly symmetric column of constant cross section this results in the following:

$$EI_y\,\partial^4(u - A)/\partial z^4 + \partial^2 M_{\bar{y}}/\partial z^2 = -m\partial^2 u/\partial t^2 \tag{3a}$$

$$EI_x\,\partial^4(v - B)/\partial z^4 - \partial^2 M_{\bar{x}}/\partial z^2 = - m\partial^2 v/\partial t^2 \tag{3b}$$

In a similar fashion Eq. (2c) is converted to an equation of torsional motion by differentiating once and adding the torsional inertia term resulting in

$$EC_w\frac{\partial^4(\varphi - C)}{\partial z^4} - GJ_e\frac{\partial^2(\varphi - C)}{\partial z^2} + \frac{PJ_0}{A_s}\frac{\partial^2\varphi}{\partial z^2} - \frac{\partial M_{\bar{z}}}{\partial_z} = -\frac{mJ_0}{A_s}\frac{\partial^2\varphi}{\partial t^2} \tag{4}$$

Substituting Eq. (1) into Eqs. (3) and (4) gives a set of partial differential equations which governs the motion of a column subjected to an axial loading only. These equations reveal the existence of nonlinear terms of two general types. First are the products of a derivative of a slope and a displacement; the second consists of the products of slopes which are sometimes neglected. The retention of these terms results in a decrease in coupling of the twisting and bending modes, and yet this coupling is shown to cause some possible destabilizing effects. The resulting equations are:

$$EI_y\frac{\partial^4(u - A)}{\partial z^4} + P\left[\frac{\partial^2 u}{\partial z^2} + v\frac{\partial^2\varphi}{\partial z^2} + \varphi\frac{\partial^2 v}{\partial z^2}\right] + 2P\frac{\partial v}{\partial z}\frac{\partial\varphi}{\partial z} + m\frac{\partial^2 u}{\partial t^2} = 0$$

$$EI_x\frac{\partial^4(v - B)}{\partial z^4} + P\left[\frac{\partial^2 v}{\partial z^2} - u\frac{\partial^2\varphi}{\partial z^2} - \varphi\frac{\partial^2 u}{\partial z^2}\right] - 2P\frac{\partial u}{\partial z}\frac{\partial\varphi}{\partial z} + m\frac{\partial^2 v}{\partial t^2} = 0 \tag{5}$$

$$EC_w\frac{\partial^4(\varphi - C)}{\partial z^4} - GJ_e\frac{\partial^2(\varphi - C)}{\partial z^2} + P\left[\frac{J_0}{A_s}\frac{\partial^2\varphi}{\partial z^2} + v\frac{\partial^2 u}{\partial z^2} - u\frac{\partial^2 v}{\partial z^2}\right] + m\frac{J_0}{A_s}\frac{\partial^2\varphi}{\partial t^2} = 0$$

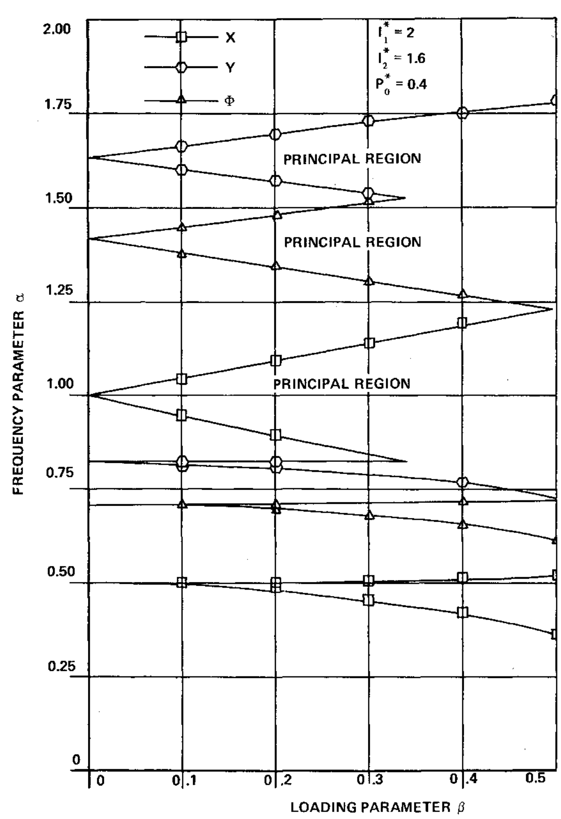

Fig. 2 Unstable regions for coupled Mathieu equations.

Simply Supported Column

The case of a column simply supported at both ends and loaded axially is considered. The boundary conditions are

$$u(0) = v(0) = \varphi(0) = u(L) = v(L) = \varphi(L) = 0$$
$$\frac{\partial^2 u(0)}{\partial z^2} = \frac{\partial^2 v(0)}{\partial z^2} = \frac{\partial^2 \varphi(0)}{\partial z^2} = \frac{\partial^2 u(L)}{\partial z^2} = \frac{\partial^2 v(L)}{\partial z^2} = \frac{\partial^2 \varphi(L)}{\partial z^2} = 0 \quad (6)$$

An approximate solution of Eqs. (5) is investigated which utilizes product forms of the temporal modes and fundamental spatial modes as follows:

$$u = r_x X \sin\pi z/L, \quad v = r_y Y \sin\pi z/L, \quad \varphi = \Phi \sin\pi z/L \quad (7)$$

An exact analysis would include the contributions of all spatial modes. However, the problem treated herein considers only the fundamental modes for bending and twisting deformations to remain consistent with investigations of inplane bending. Nearly all investigations for in-plane bending have disregarded spatial modes higher than the fundamental mode, and Bolotin[1] states that if the longitudinal force does not greatly exceed the Euler buckling load, the elastic curve differs only slightly from the first spatial mode of the linear problem. Experimental results[4,10] seem to substantiate the above statement. Similarly, the initial crookedness can be expressed as

$$A = r_x a \sin\pi z/L, \quad B = r_y b \sin\pi z/L, \quad C = c \sin\pi z/L \quad (8)$$

The initial imperfections in the column are, in general, arbitrary curves which may be defined by Fourier sine series. However, for this investigation the first terms of the sine series are assumed to be adequate to define the crookedness. Substitution of Eqs. (7) and (8) into Eqs. (5) and replacing $\sin^2\pi z/L$ and $\cos^2\pi z/L$ by the first term of their respective Fourier series results in the following set of coupled ordinary differential equations

$$(1/\omega_1^2)\ddot{X} + (1 - P^*)X - (8P^*/3\pi)(I_1^*)^{1/2}Y\Phi = a$$
$$(1/\omega_1^2)\ddot{Y} + (I_1^* - P^*)Y + (8P^*/3\pi)(I_1^*)^{-1/2}X\Phi = I_1^*b \quad (9)$$
$$(1/\omega_1^2)\ddot{\Phi} + (I_2^* - P^*)\Phi = I_2^*c$$

in which the dots refer to differentiation with respect to time t. Solutions are obtained on an analog computer (EAI TR-48) and include amplitude time responses, location of the unstable regions, and midpoint displacement traces.

Solutions

Thin-walled members with doubly symmetric cross sections are in common use as structural columns. The I-beam is a typical example. Bolotin has derived equations which define approximate boundaries for the three principal regions of instability for the case of a straight column with an unsymmetrical cross section using linear analysis. Herein, the problem of a slightly crooked and twisted column is investigated, and the effect of previously disregarded geometric nonlinearities is shown to be important under certain conditions. The column supports a constant load and is subjected to a harmonic overload. The combined load acts along a line through the centroids of the end cross sections and is given by

$$P = P_0 + P_1 \sin\theta t \quad (10)$$

Substituting Eq. (10) into Eqs. (9) and substituting new

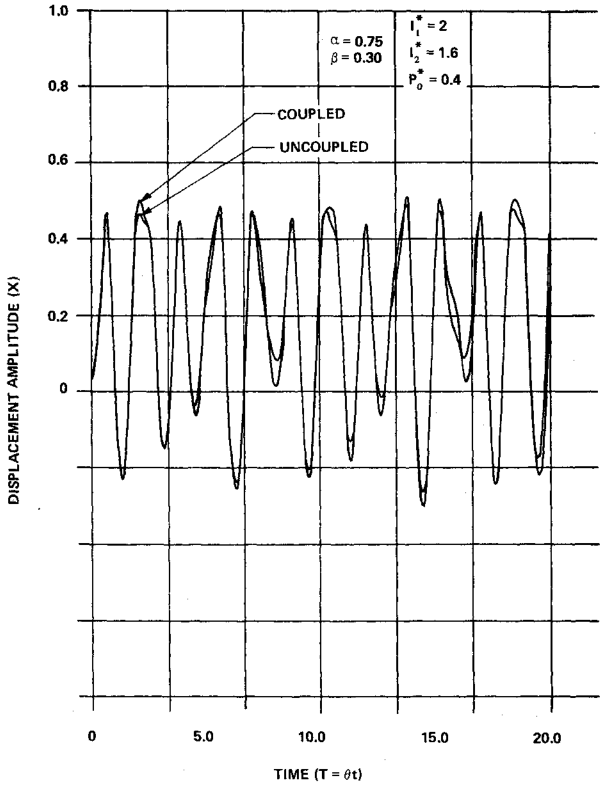

Fig. 3 Response curves for coupled and uncoupled conditions; $a = 0.25$, $b = 0.177$, $c = 0.04$, $P_0^* = 0.4$.

forms of the parameters results in Mathieu type equations

$$X'' + \frac{1}{4\alpha^2}[1 - 2\beta \sin T]X - \frac{2(I_1^*)^{1/2}}{3\pi\alpha^2} \times \left[\frac{P_0^*}{1 - P_0^*} + 2\beta \sin T\right]Y\Phi = \frac{a}{4\alpha^2(1 - P_0^*)}$$

$$Y'' + \frac{1}{4\alpha^2}\left[\frac{I_1^* - P_0^*}{1 - P_0^*} - 2\beta \sin T\right]Y + \frac{2(I_1^*)^{-1/2}}{3\pi\alpha^2} \times \left[\frac{P_0^*}{1 - P_0^*} + 2\beta \sin T\right]X\Phi = \frac{I_1^* b}{4\alpha^2(1 - P_0^*)} \quad (11)$$

$$\Phi'' + \frac{1}{4\alpha^2}\left[\frac{I_2^* - P_0^*}{1 - P_0^*} - 2\beta \sin T\right]\Phi = \frac{I_2^* c}{4\alpha^2(1 - P_0^*)}$$

in which the primes refer to differentiation with respect to T. The nonlinear terms in the above equations show that the twisting motion affects the bending vibrations, but the reverse is not true. This is due to the choice of boundary conditions. This coupling does not appear in previous works which treat the problem as three distinct motions defined by three uncoupled Mathieu equations.

Consider the solution of Eqs. (11) for a column with a principal bending stiffness ratio of two. The specified values of the parameters are

$$I_1^* = 2 \quad I_2^* = 1.6, \quad P_0^* = 0.4, \quad a = 0.25, \quad b = 0.177, \quad c = 0.04 \quad (12)$$

which indicates a low relative torsional stiffness, I_2^*. In order to determine the effects of the nonlinear coupling, the uncoupled set of equations (obtained by omitting the nonlinear terms) are considered. The left hand side of each of these equations is the well known Mathieu equation. A solution is defined as stable if the amplitudes of the solution are bounded. It should be noted that the stability regions are unaffected by the nonzero right side of these equations. The solutions of the Mathieu equation can be described as oscillatory, and depend on the values of α and β. Generally, solutions of the Mathieu

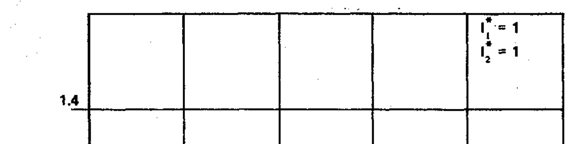

Fig. 4 Unstable regions for Mathieu equation.

equation are inconvenient to find, but it is shown in the literature that regions in the α-β plane in which the amplitudes are unbounded are separated from the regions of bounded solutions by lines on which periodic solutions of periods 2π and 4π exist. Thus the conditions under which the Mathieu

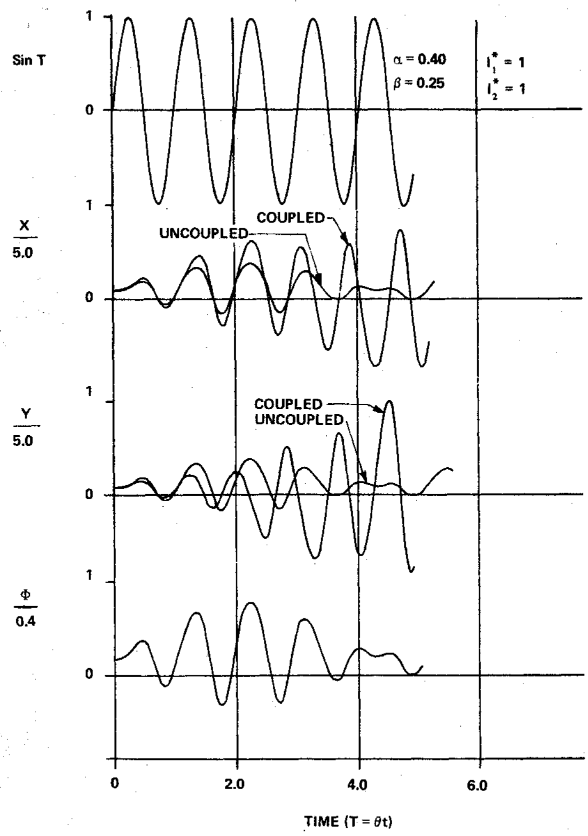

Fig. 5 Response curves for coupled and uncoupled conditions; $a = b = 0.25$, $c = 0.04$, $P_0^* = 0.4$.

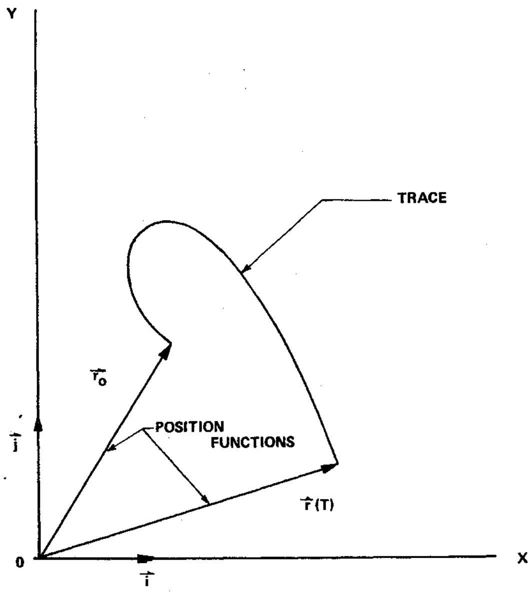

Fig. 6 Midspan trace vector.

equations has periodic solutions of period 2π or 4π, yield equations which define the boundaries of the unstable regions. This problem has been well documented[1,6,8,9] and the first and second unstable regions corresponding to the linear uncoupled equations are shown in Fig. 2.

The unstable regions corresponding to the nonlinear equations, Eqs. (11), are located using an analog computer technique developed by the authors.[2] The results indicate that the location of each unstable region is virtually the same as the uncoupled case. Moreover, the computer solutions show that there is only a slight change in the stable oscillations due to the nonlinear effects. A typical example is shown in Fig. 3. The conclusion is that for the set of parameters given in Eqs. (12) the geometric nonlinearities can be neglected, and the unstable regions can be obtained from the uncoupled Mathieu equations. However, the designer must be cautioned not to conclude that for all cases, the coupling terms and hence the geometric nonlinearities are insignificant. This will be evident from the discussion which follows.

It is logical to attempt to minimize the unstable region in the α-β plane by adjusting the parameters of the column to make the three uncoupled equations identical. This can be accomplished by setting $I_1^* = I_2^* = 1$ (See Fig. 4). But, in this case, the computer solutions of Eqs. (11) indicate that virtually every point in the α-β plane is unstable.

Typically, it can be seen in Fig. 5 that the coupled amplitudes increase with time and, therefore, by our previous definition are unstable. Such a result is difficult to accept and when one reconsiders the assumptions used in the derivation of the equations, the lack of damping is identified as the cause. Indeed, when linear viscous damping is introduced, the amplitudes of the bending modes remain bounded and the unstable regions corresponding to the uncoupled set of equations appear in the α-β plane as shown in Fig. 4. In a similar manner, Kovalenko[7] in 1950 explained the paradox of instability for

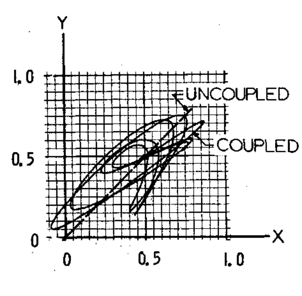

Fig. 7 Midspan trace; $\alpha = 0.7332$, $\beta = 0.25$, $I_1^* = I_2^* = 1.0$, $a = b = 0.25$, $c = 0.04$, $P_0^* = 0.4$.

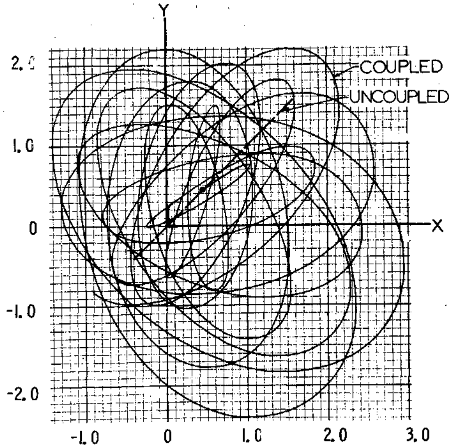

Fig. 8 Midspan trace; $\alpha = 0.4182, \beta = 0.25, I_1^* = I_2^* = 1.0, a = b = 0.25, c = 0.04, P_0^* = 0.4$.

arbitrarily small values of the loading parameter β. See Fig. 4 for the first three unstable regions for the Mathieu equation. The initial conditions shown in Fig. 5 can be obtained by solving Eqs. (11) simultaneously after setting $\overset{\prime\prime}{X} = \overset{\prime\prime}{Y} = 0$.

The most descriptive way of showing the significance of the geometric nonlinearities (coupling effects) is to plot the amplitude of one bending mode vs the amplitude of the other bending mode. Such a presentation is called a midpoint displacement trace and represents the displacement of the centroid of the midpoint of the column as seen by an observer stationed directly above the column. In Fig. 6 $\bar{r}_0$ is the initial position vector to the centroid of the column at midspan in the XY plane and defines the initial crookedness of the column. To illustrate, r_x and r_y are taken as unity and the position vector at any time is defined by

$$\bar{r} = X\bar{\imath} + Y\bar{\jmath} \tag{13}$$

Figures 7 and 8 are typical midpoint displacement traces. The effects of coupling are clearly evident and are seen to be significant. It should be noted that the linear analysis predicts bending motion in a single plane for the case of $I_1^* = I_2^* = 1$, as is shown.

Conclusion

A thin-walled column is characterized by the fact that each of its three dimensions is of a different order of magnitude. Thus the dimensions of the cross section are small when compared to the length of the column but large when compared to the thickness of the walls of the column. The nonlinear equations describing the motion of thin-walled initially crooked columns subjected to a longitudinal harmonic load were developed, and geometric nonlinearities result in the coupling of the two transverse bending modes of those columns with doubly symmetric cross sections. The coupling is provided by the twisting motion and is not predicted by the linear theory used by previous investigators. It is found that the boundaries of the unstable regions are not significantly affected by the nonlinear terms and the unstable boundaries can be obtained using only the Mathieu terms of the three equations. For the case where the bending stiffnesses and torsional stiffness are equal the geometric nonlinearities significantly affect the response of the column and should be included in the analysis. However, the coupling effect soon becomes minor as the relative stiffnesses diverge.

References

[1] Boltin, V. V., "Dynamic Stability of Straight Rods," *The Dynamic Stability of Elastic Systems*, translated from the Russian by Weingarten, Greszczuk, Trirogoff and Gallegos, Holden-Day Inc., San Francisco, 1964, pp. 281–304.

[2] Ebner, S. G. and Moody, M. L., "Parametric Response of Monosymmetric Imperfect Thin-Walled Columns under Sinusoidal Loading," *The Shock and Vibration Bulletin*, Bulletin 41, Pt. 6, Dec. 70, pp. 1–8.

[3] Evan-Iwanowski, R. M., "On the Parametric Response of Structures," *Applied Mechanics Review*, Vol. 18, Sept. 1965, pp. 699–702.

[4] Evensen, H. A. and Evan-Iwanowski, R. M., "Effects of Longitudinal Inertia Upon the Parametric Response of Elastic Columns," *Journal of Applied Mechanics*, March 1966, pp. 141–148.

[5] Gere, J. M. and Lin, Y. K., "Coupled Vibrations of Thin-Walled Beams of Open Cross Section," *Journal of Applied Mechanics*, Sept. 1958, pp. 373–378.

[6] Hayashi, Chihiro, *Nonlinear Oscillations in Physical Systems*, McGraw-Hill, New York, 1964.

[7] Kovalenko, K. R., Summary of the Doctoral dissertation, 1950, Odeskii Institut Morskogo Flota.

[8] Lubkin, S., and Stoker, J. J., "Stability of Columns and Strings Under Periodically Varying Forces, *Quarterly Applied Mathematics*, Vol. 1, 1943, pp. 215–236.

[9] McLachlan, N. W., *Theory and Applications of Mathieu Functions*, Oxford University Press, 1951.

[10] Somerset and Evan-Iwanowski, "Experiments on Parametric Instability of Columns," *Developments in Theoretical and Applied Mechanics*, Proceedings of the Second Southeastern Conference on Theoretical and Applied Mechanics, Vol. 2, Pergamon Press, 1965.

[11] Vlasov, V. Z., *Thin-Walled Elastic Beams*, translated from the Russian, by Y. Schechtman National Science Foundation OTS 61-11400, Washington, D. C., 1961.